HOW TO INVEST MORE EFFECTIVELY USING DAILY RISK MODELS: THE APT GLOBAL MODEL

AN APT WHITEPAPER

www.sungard.com/apt/learnmore

How to Invest More Effectively Using Daily Risk Models: The APT Daily Global Model

APT SOLUTIONS

- Portfolio Construction
- Portfolio Optimization
- Risk Reporting
- Multi-Factor Models
- Multi-Asset Class Coverage

Executive Summary

The APT daily model is a statistical factor model for global equities that aims to capture the systematic co-variation of daily asset returns as accurately as possible so as to provide robust estimates of market risk for global portfolios. Those risk measures may be second-moment measures (volatility, tracking error) or downside measures (VaR, CVaR and other tail risk measures). This model was first released in 2005.

One of the issues in the construction of a risk model based on daily market data is the asynchronous nature of closing prices across markets in differing time zones. For example, at 16h00 EST (4.00pm New York), there is an "implied" value of the Nikkei 225 equity index that will, in general, depend on the day's S&P500 return (amongst other things). Any risk model that does not account for such effects will almost certainly underestimate the true economic risk in international equity portfolios.

In this document, we describe the construction of the APT daily model and the calculation of a correction, based on observed systematic serial correlations, to account for the higher correlations observed across markets in lower-periodicity data. Test data for the period 2006 to 2010 is presented which demonstrates the effect of the synchronicity correction and the performance of the model over the last four years.

Table of Contents

1 The APT Statistical Factor Model

The APT factor model aims to capture the returns of assets through a set of statistical factors or components, extracted purely from historical price data. These factors are the drivers behind asset returns and are responsible for the co-variation across different assets.

Mathematically, the model is stated as

$$r_{it} = \sum_{k=1}^{K} \beta_k F_{kt} + \varepsilon_{it} \tag{1.1}$$

where r_{it} are the returns to the asset, F_t and ε_{it} are the common factor and asset-specific returns, respectively, and β_k are the factor sensitivities.

Statistical factors are usually extracted such that they are uncorrelated with each other, i.e.

$$\mathrm{E}(F_k F_{k'}) \sim \delta_{kk'} \tag{1.2}$$

where $\delta_{kk'} = 1$ if $k = k'$, and is equal to 0 otherwise. Moreover, if the factors are assumed to capture all systematic behaviour, then two further conditions will also be met:

$$\mathrm{E}(F_k \varepsilon_i) = 0$$

$$\mathrm{E}(\varepsilon_i \varepsilon_j) \sim \delta_{ij} \tag{1.3}$$

that is common factor and asset-specific returns are uncorrelated and the asset-specific returns across different assets are also uncorrelated. Therefore to test the sufficiency of a factor model, we will need to ensure that at least conditions (1.3) are met.

2 Determining the Number of Factors

An important task in factor model estimation is to know how many factors to extract. Too few or too many will result in biased estimates of the co-variation amongst asset returns, and in turn, to biased estimates of risk. To determine the number of factors in our models, we will borrow from the work on random matrices[1].

2.1 Random matrix theory

According to random matrix theory, the eigenvalues of a random correlation matrix cannot exceed a certain threshold λ_+. This suggests that by analyzing the eigenvalues of the corresponding empirical correlation matrix and comparing them to λ_+, we may test whether or not the correlations are genuine. Genuine correlations give rise to one or more eigenvalues above the maximum threshold λ_+, while those attributed to noise will have all eigenvalues below λ_+. Further, the number of eigenvalues exceeding λ_+ usually gives an indication of the number systematic factors that are responsible for genuine correlations.

To put the above on a more formal footing, let us introduce a $T \times N$ matrix of standardized returns Z, where N is the number of assets and T is the number of returns observations. From this matrix, we compute the correlation matrix as

$$P = \frac{1}{T} Z'Z \tag{2.1}$$

where the off-diagonal elements of P describe the correlations amongst the assets, while the diagonal elements are all equal to unity by construction.

The estimation of P is beset by the fact that the time series is of finite length, resulting in measurement noise. In other words, the empirical correlation matrix contains an inherent element of randomness. To distinguish genuine correlations from those that may be attributed to noise, we will need to examine the properties of the empirical correlation matrix and compare them to the corresponding random correlation matrix.

More formally, let us assume that Z is a random matrix of standardized returns. Then the theory of random matrices states that as $N \to \infty, T \to \infty$, such that $Q = T / N$ is fixed, the eigenvalues λ of the corresponding random correlation matrix P are distributed as[1]

$$F(\lambda) = \frac{Q}{2\pi} \frac{\sqrt{(\lambda_+ - \lambda)(\lambda - \lambda_-)}}{\lambda}, \quad \lambda_- < \lambda < \lambda_+ \tag{2.2}$$

where the minimum and maximum eigenvalues are given by

$$\lambda_\pm = 1 + \frac{1}{Q} \pm 2\sqrt{\frac{1}{Q}} \tag{2.3}$$

Therefore the eigenvalues of a random correlation matrix are bounded below and above by $[\lambda_-, \lambda_+]$. Eigenvalues outside this range will suggest that the correlation matrix may not be random. In particular, the observation of one or more eigenvalues above the maximum threshold λ_+ usually indicates that there are systematic factors at play.

---

[1] F.J. Dyson, Revista Mexicana de Fisica, **20**, (1971); A.M. Sengupta and P.P. Mitra, Phys. Rev. E, **60** (1999).

2.2 Number of factors in the daily global model

The number of assets for factor estimation in the daily global model as of 2010-10-06 is 8181, and the number of daily close-to-close returns used is four and half years, or 1044 daily observations. With these numbers, we compute the upper and lower bound eigenvalues of a purely random correlation matrix to be $\lambda_+ = 14.43$ and $\lambda_- = 3.24$, respectively. The largest 100 eigenvalues of the empirical correlation matrix are listed in Table 1. Comparing them with λ_+, we see that there are about 50 eigenvalues that are greater than λ_+, suggesting that there are around 50 systematic factors[2]. This number also corresponds to the point beyond which there appears to be no significant decrease in the cross-asset specific returns correlations (see Figures 1 and 2).

	λ
1	1231.01
2	602.64
3	259.53
4	191.50
5	135.14
6	95.54
7	76.45
8	62.61
9	52.27
10	50.58
11	43.70
12	41.97
13	38.44
14	36.14
15	34.81
16	33.12
17	30.23
18	29.50
19	28.79
20	27.11
21	25.37
22	24.48
23	23.60
24	22.81
25	21.76
26	21.20
27	20.17
28	20.07
29	19.52
30	19.19
31	18.74
32	18.39
33	18.09
34	17.82
35	17.40
36	17.12
37	16.94
38	16.71
39	16.45
40	16.11
41	16.02
42	15.73
43	15.58
44	15.43
45	15.30
46	15.24
47	15.19
48	14.82
49	14.56
50	14.25
51	14.12
52	14.10
53	13.87
54	13.81
55	13.69
56	13.59
57	13.45
58	13.41
59	13.16
60	13.11
61	13.10
62	13.04
63	12.92
64	12.85
65	12.69
66	12.61
67	12.51
68	12.49
69	12.30
70	12.21
71	12.20
72	12.10
73	11.99
74	11.95
75	11.79
76	11.70
77	11.66
78	11.59
79	11.55
80	11.52
81	11.43
82	11.37
83	11.34
84	11.24
85	11.18
86	11.14
87	11.11
88	11.00
89	10.96
90	10.92
91	10.83
92	10.81
93	10.78
94	10.72
95	10.63
96	10.54
97	10.53
98	10.49
99	10.43
100	10.42

Table 1. *Eigenvalues of the empirical correlation matrix for the factor estimation universe in the daily global model as of 2010-10-06. The factor estimation universe consists of 8,181 assets across all countries, with each asset containing a total of 1044 historical daily close-to-close return observations.*

---

[2] We performed a similar analysis on the APT weekly global model and found that there are 20 systematic factors.

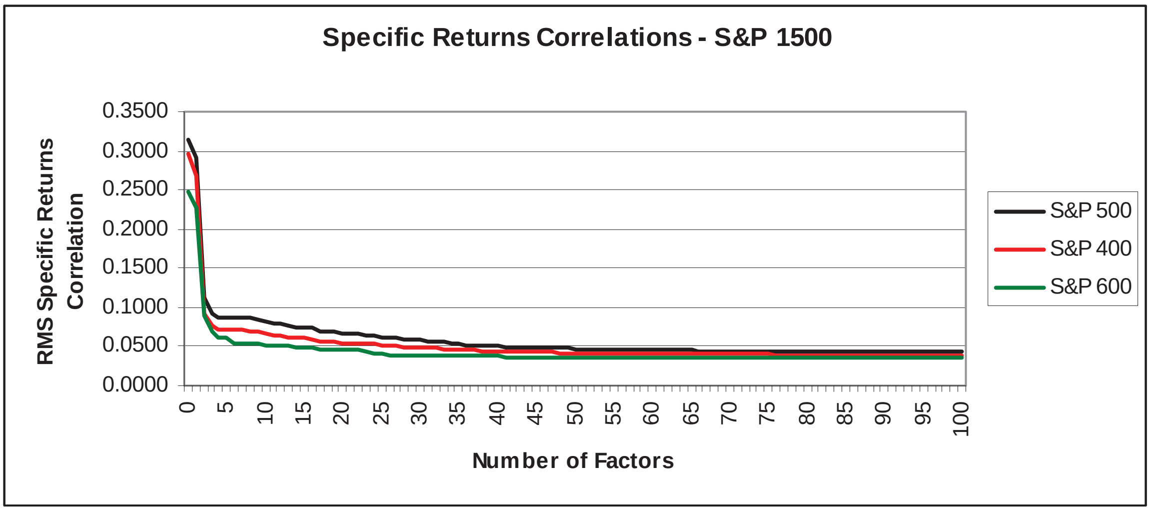

Figure 1. *Root-mean-square (RMS) specific returns correlations amongst stocks in the S&P 1500, grouped into three market capitalization buckets, with S&P 500 representing large caps, S&P 400 representing mid-caps, and S&P 600 representing small caps.*

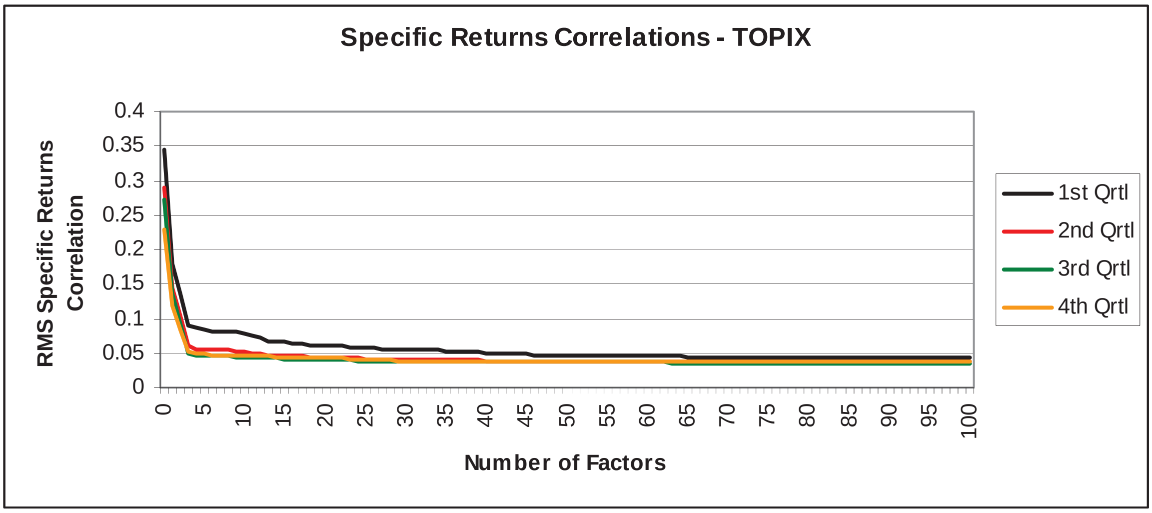

Figure 2. *Root-mean-square (RMS) specific returns correlations amongst stocks in the TOPIX, grouped into four market capitalization buckets, with the first quartile containing stocks with the largest market capitalization, and the fourth quartile, as stocks with the smallest market capitalization.*

3 Adjusting for the asynchronous nature of daily closing prices

The very nature of the different geographic time zones across the world's capital markets means that closing asset prices are often collected asynchronously. For example, closing prices in the US will always lag behind those in Japan, simply because the American market opens after the Japanese market closes.

Many studies have demonstrated the existence of spillover effects, that is, events in one market affecting those in another. In relation to the US and Japanese markets, it is conceivable that events in the Japanese market on the day before may affect asset prices in the US market when it opens. Conversely, events in the US market may affect the Japanese market when it re-opens the next day. Therefore a lead/lag relation exists between asset prices across the two markets, leading to serial or cross-correlation of daily asset prices. Failure to take this account will typically lead to biased estimates of risk, especially in portfolios containing an international mix of stocks.

To demonstrate the extent of serial correlation in stock returns data, we computed the serial correlations between daily returns of the SPX and NKY over the period 2006-10-06 to 2010-10-06. Table 1 shows the results. Clearly apparent is the significance of the cross-correlation of lag order one between US and Japanese daily equity returns.

	ρ	s.e.
-5	0.040	0.031
-4	0.004	0.031
-3	-0.031	0.031
-2	0.031	0.031
-1	0.572	0.031
0	-0.010	0.031
1	-0.077	0.031
2	-0.018	0.031
3	0.057	0.031
4	-0.065	0.031
5	0.043	0.031

Table 1. *Cross-correlation,* $\rho(R_t^{US}, R_{t-k}^{JP})$*, at different leads and lags between daily returns of the SPX and NKY over the period 2006-10-06 to 2010-10-06.*

3.1 Consequences of asynchronous daily closing prices

In the previous section, we established that there are significant cross-correlations at lag order one between daily closing returns of US and Japanese equities. The consequences of this are two fold. First, asset correlations computed using contemporaneous closing prices would understate the true underlying asset correlations. Second, the presence of cross-correlation at various leads and lags cannot be ignored when extrapolating a daily model to estimate risk forecasts over a longer horizon, for example, when using a daily model to estimate weekly risk estimates.

To gain an understanding of the second point above, suppose we have T weekly returns R_{it} for each stock i,

$$\begin{array}{l} R_{11},\ldots,R_{1T} \\ \ldots \\ R_{N1},\ldots,R_{NT} \end{array} \tag{3.1.1}$$

where each weekly return R_{it} may be expressed in terms of daily returns $r_{i,tj}$ as:

$$\begin{aligned} R_{1t} &= r_{1,t1} + \cdots + r_{1,t5} \\ &\cdots \\ R_{Nt} &= r_{N,t1} + \cdots + r_{N,t5} \end{aligned} \tag{3.1.2}$$

Here $r_{i,tj}$ denotes the daily return for stock i on day j of week t.

Let R_t be the week t return on a portfolio composed of N assets,

$$R_t = \sum_{i=1}^{N} w_i R_{it} \tag{3.1.3}$$

By definition, the variance of R_t is given[3] by

$$\sigma^2 = \frac{1}{T} \sum_{t=1}^{T} R_t^2 \tag{3.1.4}$$

Using Eqns. (3.1.2) and (3.1.3), we find that this equation may be expressed as

$$\begin{aligned} \sigma^2 = &\sum_{i=1}^{N} w_i^2 \sum_{j=1}^{5} \sigma_{ii,jj} + \sum_{i \neq i'}^{N} w_i w_{i'} \sum_{j=1}^{5} \sigma_{ii',jj} \\ &+ \sum_{i=1}^{N} w_i^2 \sum_{j \neq j'}^{5} \sigma_{ii,jj'} + \sum_{i \neq i'}^{N} w_i w_{i'} \sum_{j \neq j'}^{5} \sigma_{ii',jj'} \end{aligned} \tag{3.1.5}$$

where we have defined

$$\sigma_{ii',jj'} = \frac{1}{T} \sum_{t=1}^{T} r_{i,tj} r_{i',tj'} \tag{3.1.6}$$

---

[3] For simplicity we ignore all means, which make only a marginal difference to the end results.

Let us explain what the various terms in Eqn. (3.1.5) mean. The sum of the first two terms is simply the weekly volatility, assuming daily returns are serially uncorrelated. The third term represents the sum of contributions from autocorrelations from each stock, while the final term represents the contributions from serial correlations amongst different stocks (i.e. the lagged correlations between, e.g. US and Japanese stocks). It is clear that ignoring the final two terms may result in a biased estimate of the weekly volatility of portfolio returns.

To illustrate the above, let us randomly select an equally-weighted portfolio of 100 stocks, with fifty taken from the Russell 3000 and fifty taken from the TOPIX. Daily closing returns are observed from 2006-10-09 to 2010-10-01, representing a total of 1040 daily returns per stock. The results in Exhibit 1 are for ten randomly selected portfolios. Each table shows three columns of information. Column I shows the five-day volatility of returns (expressed as an annual percentage) assuming serial independence of daily returns; Column II shows the five-day volatility of returns taking into account autocorrelations in daily stock returns; and Column III shows the five-day volatility of returns taking into account autocorrelations and serial correlations. From the tables, it is clear that autocorrelations have a small negative impact, while serial correlations (where returns of one stock lags or leads those of another stock) have a significant positive impact. It is therefore important that one should not adapt a high frequency global model (e.g. a daily model) to one of a lower frequency (e.g. a monthly model) without adjusting for serial correlation.

Volaltility of 5-day returns - WEEKLY			
	I	II	III
1	20.69%	20.64%	22.16%
2	22.04%	21.96%	23.45%
3	20.99%	20.92%	23.32%
4	20.40%	20.33%	22.86%
5	21.21%	21.15%	22.99%
6	21.06%	21.01%	23.48%
7	19.86%	19.76%	21.35%
8	19.80%	19.72%	21.17%
9	20.37%	20.30%	23.05%
10	20.75%	20.67%	22.76%

Table 2: *Volatilities (deduced from daily returns) of randomly selected, equally weighted portfolios of 50 US stocks and 50 Japanese stocks. Column I assumes daily returns are serially uncorrelated, while columns II and III include the effects of autocorrelations, and autocorrelations and serial correlation across different stocks.*

As a numerical example of the synchronicity adjustment, we illustrate in Tables 3, 4, and 5 the synchronicity-adjusted correlations of market indices from the indices listed below. We see that, without the synchronicity adjustments, markets that are geographically far apart (e.g. SPX and the NKY) tend to have low correlations with each other. Applying the synchronicity adjustments, we see a much closer agreement with those computed using weekly returns data (in which the effects of synchronicity are much reduced). As one would expect from inspection of Table 2, the first few lags and leads are sufficient to account for the synchronicity of daily closing prices.

AMERICAS	
SPX	US S&P 500
RUT	US RUSSELL 2000
IBOV	BRAZIL IBOVESPA
MEXBOL	MEXICO (UTD MEX ST) IPC
EUROPE	
MERVAL	UK FTSE 100
UKX	GERMANY DAX
DAX	FRANCE CAC 40
CAC	ITALY MIB 30
ASIA	
NKY	JAPAN NIKKEI AVERAGE
TOPIX	JAPAN TOPIX
HIS	HONG KONG HANG SENG
SHCOMP	CHINA SHANGHAI COMPOSITE
AS30	AUSTRALIA ASX

CORRELATIONS COMPUTED FROM CONTEMPORAENOUS DAILY RETURNS													
	SPX	RUT	IBOV	MEXBOL	MERVAL	UKX	DAX	CAC	NKY	TOPIX	HIS	SHCOMP	AS30
SPX	1.00	0.93	0.69	0.73	0.63	0.54	0.58	0.55	**-0.01**	-0.03	0.26	0.06	0.24
RUT	0.93	1.00	0.63	0.68	0.59	0.46	0.51	0.48	-0.06	-0.07	0.20	0.03	0.19
IBOV	0.69	0.63	1.00	0.82	0.75	0.72	0.71	0.72	0.13	0.11	0.43	0.20	0.49
MEXBOL	0.73	0.68	0.82	1.00	0.69	0.70	0.71	0.71	0.11	0.07	0.41	0.18	0.47
MERVAL	0.63	0.59	0.75	0.69	1.00	0.62	0.62	0.63	0.14	0.13	0.34	0.16	0.42
UKX	0.54	0.46	0.72	0.70	0.62	1.00	0.88	0.93	0.25	0.22	0.45	0.17	0.65
DAX	0.58	0.51	0.71	0.71	0.62	0.88	1.00	0.94	0.23	0.19	0.42	0.18	0.60
CAC	0.55	0.48	0.72	0.71	0.63	0.93	0.94	1.00	0.24	0.21	0.42	0.18	0.63
NKY	**-0.01**	-0.06	0.13	0.11	0.14	0.25	0.23	0.24	1.00	0.97	0.56	0.26	0.61
TOPIX	-0.03	-0.07	0.11	0.07	0.13	0.22	0.19	0.21	0.97	1.00	0.53	0.25	0.58
HIS	0.26	0.20	0.43	0.41	0.34	0.45	0.42	0.42	0.56	0.53	1.00	0.47	0.68
SHCOMP	0.06	0.03	0.20	0.18	0.16	0.17	0.18	0.18	0.26	0.25	0.47	1.00	0.31
AS30	0.24	0.19	0.49	0.47	0.42	0.65	0.60	0.63	0.61	0.58	0.68	0.31	1.00

Table 3: *Correlations computed from contemporaneous daily returns.*

CORRELATIONS COMPUTED FROM CONTEMPORAENOUS WEEKLY RETURNS													
	SPX	RUT	IBOV	MEXBOL	MERVAL	UKX	DAX	CAC	NKY	TOPIX	HIS	SHCOMP	AS30
SPX	1.00	0.94	0.80	0.87	0.74	0.83	0.84	0.83	**0.62**	0.52	0.65	0.10	0.77
RUT	0.94	1.00	0.76	0.85	0.72	0.78	0.80	0.79	0.56	0.46	0.63	0.07	0.73
IBOV	0.80	0.76	1.00	0.85	0.84	0.87	0.85	0.84	0.60	0.52	0.72	0.19	0.83
MEXBOL	0.87	0.85	0.85	1.00	0.81	0.83	0.85	0.82	0.62	0.50	0.69	0.14	0.78
MERVAL	0.74	0.72	0.84	0.81	1.00	0.81	0.83	0.82	0.59	0.52	0.72	0.22	0.83
UKX	0.83	0.78	0.87	0.83	0.81	1.00	0.91	0.93	0.66	0.57	0.72	0.13	0.86
DAX	0.84	0.80	0.85	0.85	0.83	0.91	1.00	0.96	0.67	0.54	0.71	0.16	0.83
CAC	0.83	0.79	0.84	0.82	0.82	0.93	0.96	1.00	0.68	0.59	0.71	0.16	0.85
NKY	**0.62**	0.56	0.60	0.62	0.59	0.66	0.67	0.68	1.00	0.95	0.71	0.25	0.78
TOPIX	0.52	0.46	0.52	0.50	0.52	0.57	0.54	0.59	0.95	1.00	0.64	0.24	0.70
HIS	0.65	0.63	0.72	0.69	0.72	0.72	0.71	0.71	0.71	0.64	1.00	0.36	0.81
SHCOMP	0.10	0.07	0.19	0.14	0.22	0.13	0.16	0.16	0.25	0.24	0.36	1.00	0.28
AS30	0.77	0.73	0.83	0.78	0.83	0.86	0.83	0.85	0.78	0.70	0.81	0.28	1.00

***Table 4:** Correlations computed from weekly returns.*

The weekly correlations, in general, can be significantly large for those markets, such as US and Japan, that have strong lead/lag relationships. A test of our synchronicity correction (also called the asynchroneity adjustment) will be its ability to reproduce these larger correlations in the context of daily data.

In Table 4, above, we present these correlations.

CORRELATIONS COMPUTED FROM ASYNCHRONEITY-ADJUSTED DAILY RETURNS

	SPX	RUT	IBOV	MEXBOL	MERVAL	UKX	DAX	CAC	NKY	TOPIX	HIS	SHCOMP	AS30
SPX	1.00	0.94	0.79	0.83	0.71	0.82	0.83	0.83	**0.61**	0.53	0.66	0.19	0.74
RUT	0.94	1.00	0.73	0.80	0.69	0.75	0.78	0.77	0.58	0.51	0.62	0.14	0.71
IBOV	0.79	0.73	1.00	0.83	0.83	0.83	0.81	0.82	0.59	0.55	0.73	0.30	0.80
MEXBOL	0.83	0.80	0.83	1.00	0.79	0.80	0.82	0.81	0.63	0.56	0.73	0.23	0.76
MERVAL	0.71	0.69	0.83	0.79	1.00	0.80	0.79	0.80	0.63	0.59	0.73	0.30	0.80
UKX	0.82	0.75	0.83	0.80	0.80	1.00	0.90	0.93	0.68	0.61	0.77	0.25	0.86
DAX	0.83	0.78	0.81	0.82	0.79	0.90	1.00	0.96	0.67	0.59	0.73	0.26	0.82
CAC	0.83	0.77	0.82	0.81	0.80	0.93	0.96	1.00	0.70	0.63	0.74	0.26	0.85
NKY	**0.61**	0.58	0.59	0.63	0.63	0.68	0.67	0.70	1.00	0.96	0.68	0.24	0.77
TOPIX	0.53	0.51	0.55	0.56	0.59	0.61	0.59	0.63	0.96	1.00	0.63	0.22	0.73
HIS	0.66	0.62	0.73	0.73	0.73	0.77	0.73	0.74	0.68	0.63	1.00	0.43	0.80
SHCOMP	0.19	0.14	0.30	0.23	0.30	0.25	0.26	0.26	0.24	0.22	0.43	1.00	0.30
AS30	0.74	0.71	0.80	0.76	0.80	0.86	0.82	0.85	0.77	0.73	0.80	0.30	1.00

Table 5: *Correlations computed from synchronicity-adjusted daily returns.*

In Table 5, we can see that the lead/lag corrected daily returns are much more representative of the weekly returns. As such, by employing such a model, we should be able to gain all the benefits of daily data (namely better measurements of daily component returns and increased number of observations) whilst still accounting for the fact that the raw closing prices do not fully reflect the shared risk in international portfolios.

3.2 The CHMSW covariance matrix estimator

We have shown that serial correlation at various leads and lags is an important consideration to take into account, especially in portfolios containing an international mix of stocks. Apart from non-synchronous trading across the various markets in the world, illiquidity in the markets also introduces serial correlation. To correct for this, Cohen et al[4] (CHMSW) proposed the following adjustment to the contemporaneous covariance matrix:

$$\Theta(q) = \theta_{00} + \sum_{l=1}^{q} (\theta_{t,t+l} + \theta'_{t,t+l}) \qquad (3.2.1)$$

where $\theta_{t,t+l}$ are cross-product covariances given by[5]

$$\theta_{t,t+l} = \frac{1}{T} \sum_{t'=1}^{T} R'_{t,t'} R_{t,t'+l} \qquad (3.2.2)$$

The problem with estimator given in Eqn. (3.2.1) is that the resulting covariance matrix may not be positive semi-definite. To correct for this, we introduce Bartlett kernel weights[6] d_l (guaranteeing a positive definite matrix),

$$\Theta(q) = \theta_{00} + \sum_{l=1}^{q} d_l (\theta_{t,t+l} + \theta'_{t,t+l}) \qquad (3.2.3)$$

where $d_l = 1 - l/(q+1)$.

---

[4] K. J. Cohen, G. A. Hawawini, S. F. Maier, R. A. Schwartz, and D. K. Whitcomb, *Friction in the Trading Process and the Estimation of Systematic Risk*, Journal of Financial Economics, **12** (1983).

[5] We have assumed the returns have been mean zero for simplicity.

[6] See, e.g. D. W. K. Andrews, *Heteroskedasticity and Autocorrelations Consistent Covariance Matrix Estimation*, Econometrica, **59** (1991).

3.3 Implementation of the CHMSW covariance matrix estimator within a factor model

Assuming specific returns are serially independent, the CHMSW covariance matrix estimator may be computed within a linear factor model as

$$\Sigma = B'\Sigma_F(q)B + \Psi \tag{3.3.1}$$

where B are the $k \times N$ factor loadings and Ψ is an $N \times N$ diagonal matrix of specific variances, and $\Sigma_F(q)$[7] is a $k \times k$ synchronicity-adjusted factor covariance matrix, given by

$$\Sigma_F(q) = \frac{1}{T}\sum_{t=1}^{T} F_t^{'} F_t + \sum_{l=1}^{q} d_l \left[\frac{1}{T}\sum_{t=1}^{T-l} F_t^{'} F_{t+l} + \frac{1}{T}\sum_{t=1}^{T-l} F_{t+l}^{'} F_t \right] \tag{3.3.2}$$

The synchronicity-adjusted factor covariance matrix may be spectrally decomposed as

$$\Sigma_F = \Gamma\Lambda\Gamma' \tag{3.3.3}$$

where Λ and Γ are the eigenvalues and eigenvectors of Σ_F, respectively.

Substituting Eqn. (3) into Eqn. (1), we can write the synchronicity-adjusted factor covariance matrix as

$$\Sigma = \tilde{B}'\tilde{B} + \Psi \tag{3.3.4}$$

where we have defined synchronicity-adjusted factor loadings $\tilde{B}$ as

$$\tilde{B} = \Lambda^{1/2}\Gamma' B \tag{3.3.5}$$

---

[7] In our synchronicity adjusted daily factor models, we use $q = 5$. This value gives results that are relatively consistent with correlations computed using weekly returns data.

The synchronicity-adjusted systematic covariance matrix is defined as

$$\Sigma_{sys} = \tilde{B}'\tilde{B} \tag{3.3.6}$$

Using the identities $\Gamma'\Gamma = I$ and $F'F = I$, we can re-write Eqn. (6) as

$$\Sigma_{sys} = \tilde{B}'\Gamma'F'F\Gamma B = \tilde{B}'\tilde{F}'\tilde{F}\tilde{B} \tag{3.3.7}$$

where $\tilde{F} = F\Gamma$ are the synchronicity-adjusted factor returns and $\tilde{F}\tilde{\beta}$ are the synchronicity-adjusted systematic returns.

4 Validation of the model

Out of sample back-tests employing the APT daily synchronicity-corrected factor model are employed in order to validate the calculations. We validate the model for portfolios composed of securities from major equity indices. We demonstrate, using out of sample testing, some of the issues with vanilla historical risk estimates. In particular, we find that optimized portfolios constructed employing quantitative techniques can show large, out-of-sample violations of their ex ante Historical VAR risk forecasts. The APT factor model appears to provide a methodology for dealing with this well-known issue.

4.1 What is wrong with using historical data?

Mathematically, the co-movement of a pair of assets is described by the covariance, defined as

$$\sigma_{ij} = \mathrm{E}[(r_i - \mu_i)(r_j - \mu_j)] \tag{4.1.1}$$

where r_i and r_j are the returns of assets i and j, respectively, and μ_i and μ_j are their mean values. The operator $\mathrm{E}(X)$ denotes the expectation of a random variable X .

The co-movements of all assets i and j are summarized in the following variance/covariance matrix,

$$\Sigma = \begin{pmatrix} \sigma_{11} & \cdots & \sigma_{1N} \\ \vdots & \ddots & \vdots \\ \sigma_{N1} & \cdots & \sigma_{NN} \end{pmatrix} \tag{4.1.2}$$

where N is the number of assets and the variances are represented by the diagonal elements $\sigma_{ii} > 0$. The covariance matrix represents an intricate web that links the co-movements of all assets and therefore precludes the examination of any one cell in isolation to the rest, i.e. the elements of the covariance matrix must be estimated consistently.

A consistent estimator of the covariance matrix is the sample covariance matrix, defined as

$$\Sigma_S = \frac{1}{T-1} R'R \tag{4.1.3}$$

where R is a $T \times N$ matrix of normalized asset returns, i.e. where the sample mean returns have been subtracted. The covariance matrix looks rather innocuous as it stands, but let us pause for a moment. Given N assets, the number of parameters that needs to be estimated is equal to $N(N+1)/2$, whereas the number of observations available for estimation is equal to NT . For N close to T , it is readily seen that we do not have enough observations to estimate the parameters reliably, leading to a very noisy estimate of the true covariance matrix. In the general case where $N > T$, the sample covariance matrix becomes singular, making its use in asset allocation rather problematic.

4.2 Back-testing Methodology

We perform out-of-sample tests over a sequence of thirteen models, estimated monthly from 2009-09-02 to 2010-09-01. The testing period begins on 2009-09-03 and runs to 2010-09-30.

The methodology is as follows:

1) At each model estimation date, we take a benchmark market portfolio on that date and compute its 99% and 95% VAR using the synchronicity-adjusted factor model. For comparison, we also compute the corresponding VARs using historical data[8].

2) Measure the out-of-sample P&L of the portfolio until the next model date[9].

3) Repeat 1 and 2 until the end of the back-test period.

4) Compare the distribution of VAR violations to the predictions of the models.

The parametric factor model VAR, X , at the confidence level, $1-\alpha$, is the value that solves the following equation:

$$\alpha = \frac{1}{T}\sum_{t=1}^{N} N((X - \hat{r}_t(t))/\sigma) \tag{4.2.1}$$

where $\hat{r}_t(t)$ are the systematic returns, σ is the portfolio specific risk, and N is the cumulative normal distribution.

Notice that this is a parametric (analytic) calculation that does not involve time-consuming Monte Carlo estimation techniques. Despite the use of the cumulative normal distribution the methodology is not based on the assumption of random walk-type normal market returns, as we have used our synchronicity correction to adjust market returns before modelling, and the distributions of the systematic risk factors are clearly non-Gaussian.

---

[8] For the S&P 500 portfolio we use the past 1,000 trading to estimate the historical VAR, while for the combine S&P 500 and Nikkei 225 portfolios we use the past 900 trading days (excluding both US and Japanese national holidays).

[9] The constituent weights are held constant over the out-of-sample period.

4.3 Market portfolio back-tests

4.3.1 S&P 500 portfolio

CL	Average VAR		% of daily VAR violations	
	Factor	Historical	Factor	Historical
99%	-5.43%	-5.39%	0.00%	0.00%
95%	-2.21%	-2.60%	4.43%	2.95%

Table 6. *Summary statistics for the out-of-sample performance for the S&P 500 portfolio.*

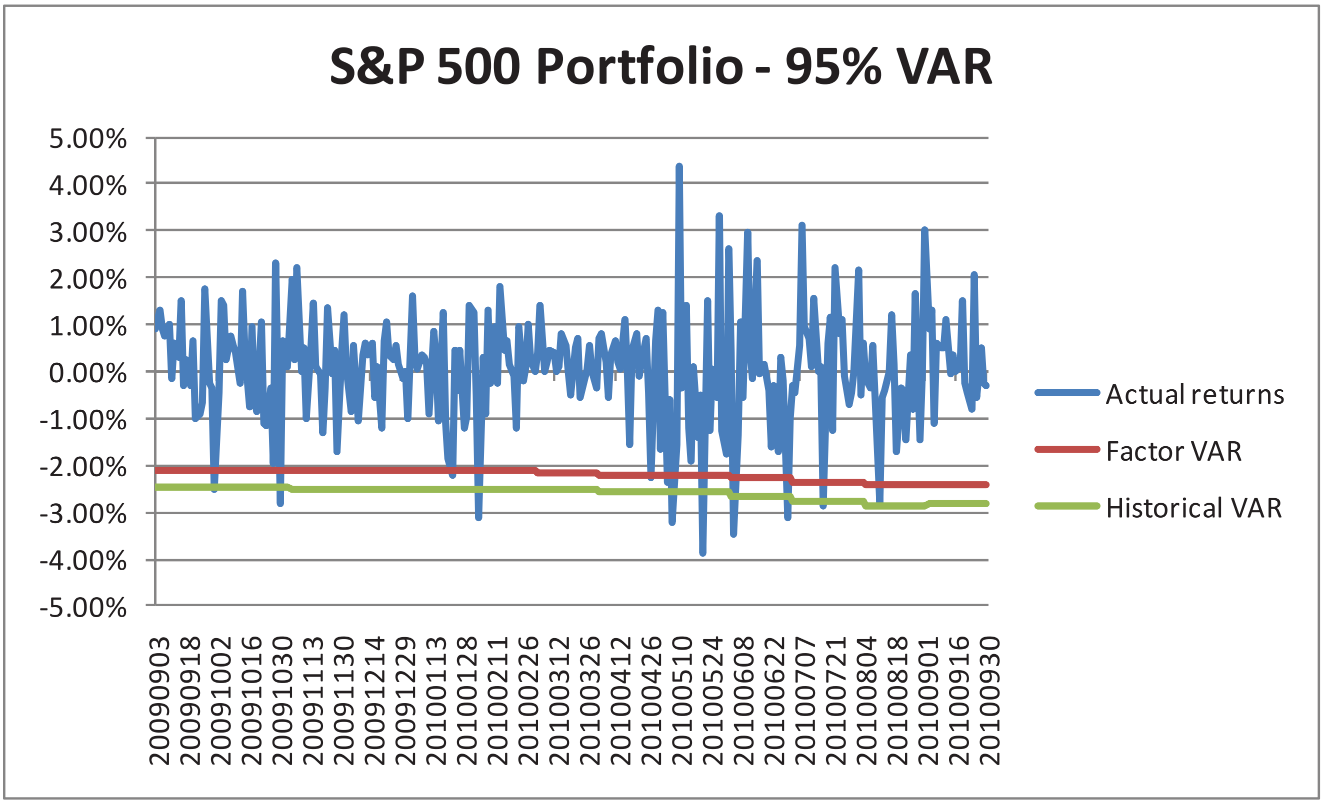

Figure 3. *Out-of-sample performance for the S&P 500 portfolio at the 95% confidence level.*

4.3.2 A combined portfolio of the S&P 500 and Nikkei 225

In this section, we validate the model based on a combined portfolio of the S&P 500 and the Nikkei 225 (held in equal proportions of 50% each). The returns and VAR of this combined portfolio is computed in USD.

CL	Average VAR		% of daily VAR violations	
	Factor	Historical	Factor	Historical
99%	-4.60%	-4.27%	0.00%	0.00%
95%	-2.10%	-1.90%	1.98%	2.77%

Table 7. *Summary statistics for the out-of-sample performance for the combined portfolio of the S&P 500 and Nikkei 225.*

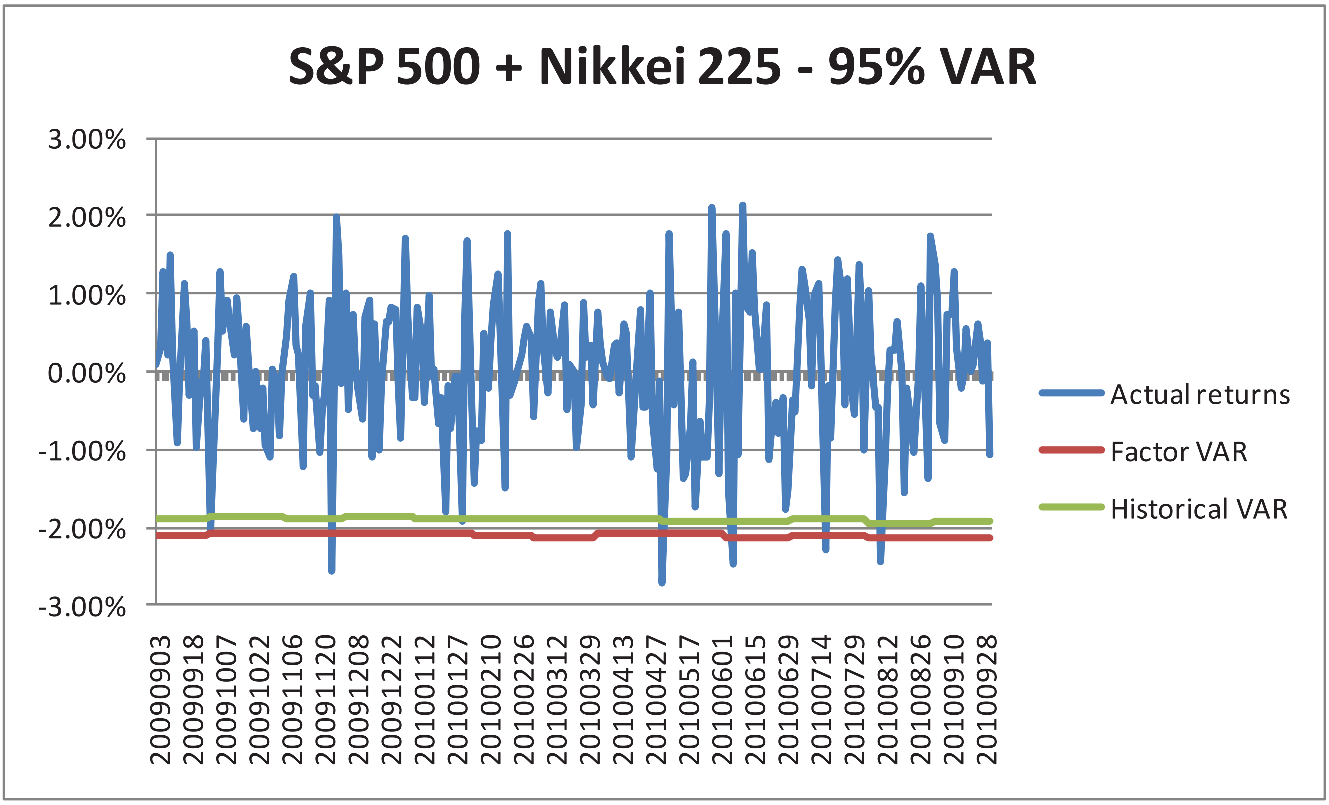

Figure 4. *Out-of-sample performance for the combined portfolio of the S&P 500 and Nikkei 225 at the 95% confidence level.*

4.4 Optimal Portfolio Back-tests

In addition to straightforward long-only portfolio testing, APT performs extensive testing employing optimal portfolios. Why so? Aside from their practical interest as estimates of the minimum variance portfolio, these portfolios provide the most stringent tests of the hedging relationships predicted by a given risk model as well as their persistence out-of-sample. In addition, these sorts of tests are particularly relevant in an environment where margin and/or trading limits are often set on the basis of raw historical VAR. In these cases, asset managers have an intrinsic motivation to minimize the apparent historical risk of their portfolios, and the sorts of techniques that we discuss herein are the most straightforward methodologies for doing so.

The methodology is as follows:

1) Build a long minimum variance portfolio[10] on 2009-09-02 using the APT asyncrhoneity-adjusted daily model. Do the same but using historical daily returns data over the past four years.

2) Compute the predicted VAR at the 99% and 95% confidence levels.

3) Measure the performance of the optimized portfolio over the out-of-sample period from 2009-09-03 to 2010-09-30 and compare it with the predicted VAR.

4.4.1 Optimized portfolio of US equities

	Out-of-sample volatility
Factor	10.95%
Historical	11.53%

CL	Predicted VAR		% of daily VAR violations	
	Factor	Historical	Factor	Historical
99%	-2.45%	-2.73%	0.37%	0.00%
95%	-1.11%	-1.42%	4.06%	2.58%

Table 8. *Out-of-sample performance statistics for the optimized long-only minimum variance portfolio of US equities.*

4.4.2 Optimized portfolio of US and Japanese equities

	Out-of-sample volatility
Factor	8.68%
Historical	8.80%

CL	Predicted VAR		% of daily VAR violations	
	Factor	Historical	Factor	Historical
99%	-2.05%	-1.94%	0.00%	0.00%
95%	-0.98%	-1.12%	3.56%	1.58%

Table 9. *Out-of-sample performance statistics for the optimized long-only minimum variance portfolio of US and Japanese equities.*

---

[10] Candidate stocks are selected from the S&P 500 as of 2010-09-02. Only stocks with full returns history over the estimation and out-of-sample period are considered.

5 Conclusions

We have described some of the issues associated with the estimation of a daily global, synchronicity-adjusted factor model for equities. These include dealing with serial correlation and validating the model out of sample. By leveraging the power of daily data, and applying econometric corrections to account for serial correlation, the APT global daily factor model succeeds in providing robust estimates which are well-suited to risk management and portfolio optimization.

SUNGARD APT

ai.solutions@sungard.com

Americas:
340 Madison Avenue
New York
NY 10173
USA
Tel: +1 646 445 1018

Middle East:
Knowledge Village
Block 2A
Dubao
United Arab Emirates
Tel: +971 4 3911180

Europe:
25 Canada Square
London
E14 5LQ
UK
Tel: +44 20 8081 2720

Asia:
71 Robinson Road
#15-01
Singapore 068895
Tel: +65 6308 8000

100 Queen's Road Central
Central
Hong Kong
Tel: +852 3719 0800
Fax: +852 2869 6272

Australia:
Level 12
9 Hunter Street
Sydney, NSW 2000
Australia

About SunGard's APT

APT provides investment technology for a broad range of asset classes, countries and regions including data and software for understanding market risk, credit risk, liquidity risk and for portfolio construction and performance analysis. APT provides investors with statistical market risk models, performance and risk analytics and portfolio optimization and construction tools. APT's customers include institutional and retail asset managers, pension funds, private wealth managers, hedge funds, broker/dealers, prime brokers and proprietary traders.

www.sungard.com/apt/learnmore

About SunGard

SunGard is one of the world's leading software and technology services companies. SunGard has more than 20,000 employees and serves 25,000 customers in 70 countries. SunGard provides software and processing solutions for financial services, higher education and the public sector. SunGard also provides disaster recovery services, managed IT services, information availability consulting services and business continuity management software. With annual revenue exceeding $5 billion, SunGard is ranked 380 on the Fortune 500 and is the largest privately held business software and IT services company.

www.sungard.com

www.sungard.com/apt/learnmore